Dynamics of cavities in a duct flow

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A model is presented for the dynamics of cavities associated with the cavitation in a duct flow. Measurements taken from cine films of cavitating flows indicate detached cavities take between three and four times longer to collapse than would be expected from Rayleigh's classical theory. To account for this, a theoretical model has been developed using the one-dimensional conservation equations of mass and momentum in the duct and Rayleigh's equation governing cavity wall motion. A numerical solution is obtained, and it is shown the cavity collapse time increases rapidly above the Rayleigh result when a duct parameter, based on maximum cavity size, duct cross-sectional area, and a flow-length scale, exceeds unity.

The predictions of the model agree very well with the measurements derived from cine films when the flow-length scale is taken to be equal to the mean length of the attached cavity.

Keywords: cavitation; duct flow; theoretical model

Introduction

The purpose of this paper is to present a model for the dynamics of cavities associated with cavitating flow confined in a duct. In developing models of cavitation erosion and noise, it may be assumed for simplicity that the presence of the duct walls does not affect the cavity growth and collapse, apart from the production of a microjet for a cavity collapsing adjacent to a solid surface. The validity of this assumption is clearly doubtful especially where large "fixed" cavities are involved. Recent observations[1,2] of cavitating flow around convergent-divergent wedge and 60° symmetrical wedge inducers in a duct have been made using high-speed cine photography. These have shown the collapse time for such cavities is between three and four times longer than predicted by Rayleigh's classical theory[3] for the collapse of a spherical cavity in an unbounded liquid. Another feature of these observations is that the local mean rate of volume change produced by the cavity collapse is a large fraction, about 10%, of the total volumetric flow rate. Both these observations indicate a significant interaction between the cavity and the duct flow.

A theoretical model originally developed to help explain these observations[4] has now been extended and modified. The model is based on the one-dimensional conservation equations of mass and momentum for the flow in the duct and, in the case of steady-state conditions, conforms with the familiar Borda-Carnot analysis. The equations were originally written in terms of an unknown cavity volume of arbitrary shape, but they cannot be solved since the downstream pressure is not known. By using Rayleigh's equation for the motion of a spherical cavity wall, the equations can be solved and yield details of the cavity wall motion and the variation of downstream pressure. Rayleigh's cavity motion is seen to be applicable when the condition $4\pi\ell_a R_m/A_2 \ll 1$ holds, where ℓ_a is a flow-length scale, R_m maximum cavity radius, and A_2 duct cross-sectional area.

Equation of motion for cavity wall

The presence of a growing or collapsing cavity in a duct implies a change in volumetric flow rate between locations upstream and downstream of the cavity. The original theory treated this case, and it was concluded that this situation could not persist indefinitely, since either the cavity would grow infinitely large and the flow would become choked, or the cavity would collapse to nothing and cavitation would cease. Observations of cavitation in venturi-type channels[1,2,5,6] indicate at some instant part of the cavity becomes detached from the main cavity and is convected downstream eventually to collapse. While the detached part is collapsing, the remaining attached cavity continues to grow until the next detachment when the whole process is repeated. This mechanism allows the growth and collapse of cavities to occur while the upstream and downstream volume flow rates remain equal. It is assumed the inertia of the fluid column both upstream and downstream of the cavitation zone is large enough to suppress short-term variations in flow rate. This assumption also allows the pressure to depart from the steady-state, that is, Borda-Carnot, value in the short term. For steady conditions over a long period, it is expected that the average pressure will be equal to the steady-state value.

If the upstream and downstream flow rates are equal, the volumetric growth rate of the attached cavity must be exactly equal numerically to the volumetric collapse rate of the detached cavity, and the total cavity volume remains constant during the motion. This motion can be modeled by a source–sink pair where the components are separated by a distance corresponding to the separation of the attached and detached cavities. An equation of motion for the volume of either the attached cavity or the detached cavity can be deduced on assuming the source–sink pair is located within a control volume enclosing the fluid from the throat to a point well downstream of the collapsing cavity where the channel walls are parallel (Figure 1). The flow into and out of the control volume is assumed to be effectively one dimensional; that is, variations in pressure and velocity across the duct may be accounted for by using the appropriate average values. The cavities are assumed to contain vapor with a negligible gas content, and the growing cavity is assumed to be attached to the cavitation inducer at the throat and to span the whole of the rear of the inducer, an assumption that holds at low enough cavitation number.

Since the strengths of the source and sink are equal and opposite, we can write

$$\dot{V} + \dot{V}' = 0 \tag{1}$$

and from the continuity equation, we obtain

$$A_2U_2-A_1U_1=0 \tag{2}$$

By applying the principle of conservation of momentum to the control volume and neglecting viscous and body forces, it can be deduced that

$$(p_1-p_v)A_1-(p_2-p_v)A_2=\rho A_2U_2^2-\rho A_1U_1^2+\rho\frac{d}{dt}\int Q\,dx \tag{3}$$

If the source and sink are assumed to be effectively located at two points separated instantaneously by a distance ℓ, then Q is equal to A_1U_1 both upstream and downstream of the source–sink pair from Eq. (2) and equal to $A_1U_1+\dot V'$ or $A_1U_1-\dot V$, using Eq. (1), between the two components. Since A_1U_1 is taken to be constant, the integral term becomes $-\rho d(\ell\dot V)/dt$ in terms of the detached cavity. This term decomposes into $-\rho\ell\ddot V-\rho U_c\dot V$, where U_c is the convection velocity of the detached cavity, strictly relative to the effective center of the attached cavity. Since the detached cavity is approximately spherical, these terms may be written as functions of instantaneous cavity radius R when it is found that the term in $\ddot V$ involves $4\pi\ell R/A_2$, and that in $\dot V$ involves $4\pi R^2/A_2$. Since in experiments[2] in a duct of 800 mm^2 cross section, ℓ is found to be generally about 90 mm, and the maximum or initial value of R about 8 mm, the latter term is at most one tenth the former and generally very much smaller still. In this case, it is appropriate to neglect the term in $\dot V$ and replace ℓ by a suitable average value, ℓ_a.

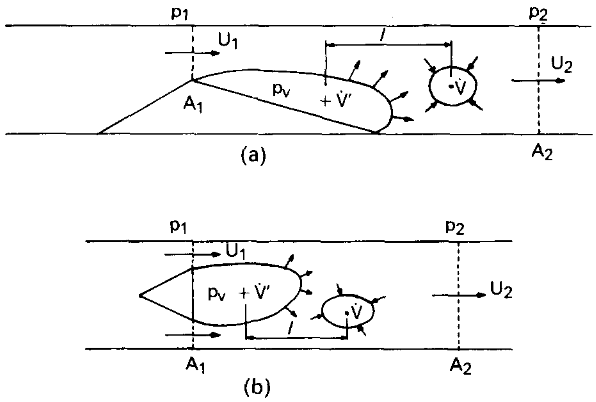

Figure 1 Sketch of control volume showing attached and detached cavities and source–sink locations for (a) convergent-divergent wedge and (b) symmetrical wedge inducers

On substituting for U_2 using Eq. (2), Eq. (3) becomes

$$\left(\frac{p_1-p_v}{\rho}\right)\frac{A_1}{A_2}-\left(\frac{p_2-p_v}{\rho}\right)+U_1^2\frac{A_1}{A_2}\left(1-\frac{A_1}{A_2}\right)+\frac{\ell_a\ddot V}{A_2}=0 \tag{4}$$

In the case of steady flow without cavities when $\ddot V=\dot V=0$, Eq. (4) reduces to a result similar to the Borda-Carnot formula for the pressure recovery in a sudden expansion. The downstream pressure p_B is given by

$$\frac{p_B-p_v}{\rho}=\left(\frac{p_1-p_v}{\rho}\right)\frac{A_1}{A_2}+U_1^2\frac{A_1}{A_2}\left(1-\frac{A_1}{A_2}\right) \tag{5}$$

It may be assumed the throat pressure p_1 is constant and equal to vapor pressure throughout the motion, since the streamline curvature is small at this point, and it is observed that pressure fluctuations in the vicinity of the throat are much reduced compared with downstream.[7] On combining Eqs. (4) and (5) then,

$$\frac{p_B-p_v}{\rho}-\left(\frac{p_2-p_v}{\rho}\right)+\frac{\ell_a\ddot V}{A_2}=0 \tag{6}$$

Since the detached cavity is presumed to be effectively spherical, the downstream pressure p_2-p_v, which is allowed to vary while the flow rate remains constant, can be related dirctly to the detached cavity radius by using Rayleigh's equation for cavity wall motion. The use of this equation requires spherical symmetry of motion in the vicinity of the cavity. This will be approximately true if the cavity is somewhat smaller than the duct size; the condition to be satisfied is $2\pi R^2/A_2\ll 1$. Adopting the same example as above, this requires the cavity radius to be substantially less than about 11 mm. Since the initial radius of most cavities is about 8 mm, the assumption of symmetry in the early stages of the motion is questionable. However, as will be seen later, this does not appear to lead to any significant errors.

Since the fluid velocity does not change downstream of the detached cavity, the driving pressure for the Rayleigh equation is taken to be the same as the pressure at the downstream station of the control volume so that

$$\frac{p_v-p_2}{\rho}=R\ddot R+\tfrac{3}{2}\dot R^2 \tag{7}$$

When p_2-p_v is eliminated from Eq. (6) by using Eq. (7) and cavity volume is written in terms of radius, the equation becomes

$$\left(1+\frac{4\pi\ell_aR}{A_2}\right)R\ddot R+\left(\frac{3}{2}+\frac{8\pi\ell_aR}{A_2}\right)\dot R^2+\frac{p_B-p_v}{\rho}=0 \tag{8}$$

Assuming initial conditions, $\dot R=0$ when $R=R_m$, this equation

Notation

- A_1 Throat cross-sectional area
- A_2 Downstream duct cross-sectional area
- k Parameter relating duct cross-sectional area, source–sink separation, and maximum cavity radius $(4\pi\ell_aR_m/A_2)$
- ℓ Effective distance between source and sink
- ℓ_a Time-averaged value of ℓ
- p_1 Static pressure at throat
- p_2 Static pressure at downstream station
- p_B Steady-state pressure at downstream station
- p_v Saturated vapor pressure at bulk liquid temperature
- p_∞ Static pressure at infinity
- Q Volumetric flow rate at any cross section
- R Instantaneous spherical cavity radius
- R_m Maximum spherical cavity radius
- t Time
- t_c Cavity collapse time
- U_1 Mean velocity at throat
- U_2 Mean velocity at downstream station
- V Volume of detached cavity
- V' Volume of attached cavity
- x Distance measured in downstream direction
- ρ Mass density

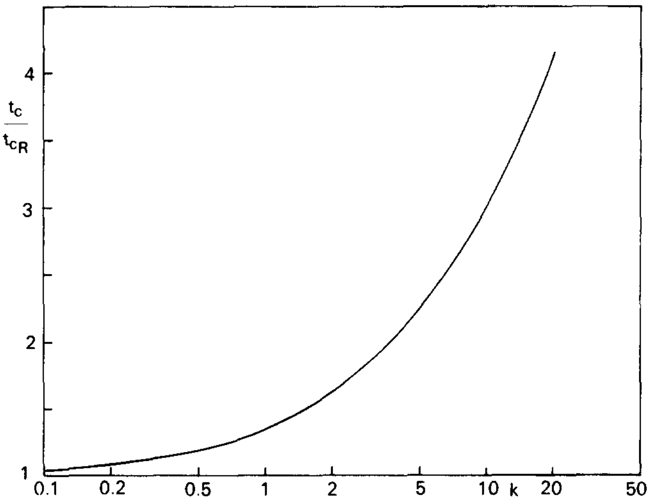

Figure 2 Total cavity collapse time t_c relative to Rayleigh's collapse time t_{cR} as a function of duct parameter k $(4\pi l_a R_m/A_2)$

may be integrated directly to give the following result:

$$\dot{R}^2=\frac{\frac{2}{3}\left(\frac{p_B-p_v}{\rho}\right)\left(\frac{R_m^3}{R^3}-1\right)}{\left(1+\frac{4\pi\ell_a R}{A_2}\right)} \tag{9}$$

The similarity with Rayleigh's original result for a cavity collapsing in an infinite medium is clear, and Eq. (9) reduces to this form when $4\pi\ell_a R/A_2 \ll 1$. This condition must apply sooner or later during the cavity collapse, thus the Rayleigh-type collapse is recovered eventually. However, if $4\pi\ell_a R/A_2$ is unity or greater, the collapse motion is modified considerably.

Cavity collapse time

The result (Eq. 9) can, in principle, be integrated to determine the variation of cavity radius with time and, hence, obtain the total time for collapse to occur. It seems to be difficult, if not impossible, to integrate Eq. (9) analytically except in the special case of a cavity in an unbounded medium ($A_2\rightarrow\infty$) between the limits $R=0$ and $R=R_m$. This integration, originally performed by Rayleigh, gives the following result for collapse time t_{cR}:

$$t_{cR}=0.91468R_m\sqrt{\frac{\rho}{p_B-p_v}} \tag{10}$$

Equation (9) can be integrated numerically in terms of a parameter k relating duct size, source–sink separation, and maximum cavity radius:

$$k=\frac{4\pi\ell_a R_m}{A_2} \tag{11}$$

The results of the numerical integration for a range of k values are shown in Figure 2, where the total collapse time is expressed as a ratio with the Rayleigh collapse time, that is, the value when k is zero. It can be seen that the collapse time is increased by a small amount for k up to about unity and is substantially increased above this value, being between three and four times the Rayleigh time for k between 10 and 20.

Variation of downstream pressure

The model can be used to calculate the variation of pressure at the downstream station (p_2-p_v), which is presumed to be the driving pressure in Rayleigh's equation (Eq. 7). By using Eqs. (7) and (8) in combination with Eq. (9), it may be shown that p_2 can be expressed relative to the Borda-Carnot recovery pressure, p_B, as follows:

$$\frac{p_2-p_B}{p_B-p_v}=\frac{\frac{4\pi\ell_a R}{3A_2}\left(\frac{R_m^3}{R^3}-4\right)-\left(\frac{4\pi\ell_a R}{A_2}\right)^2}{\left(1+\frac{4\pi\ell_a R}{A_2}\right)^2} \tag{12}$$

At the start of the motion when $R=R_m$, the pressure p_2 is somewhat below the pressure p_B, and as the collapse proceeds, the pressure increases, becomes positive, and then infinite at the moment of complete collapse. For the collapse in an infinite medium when k is zero, the pressure remains constant equal to p_B, as expected for the pressure at infinity.

Since the variation in pressure is both negative and positive, the question whether the time-averaged value of the pressure, p_2-p_B, is zero or not naturally arises. This can be investigated by numerical integration of Eq. (12) over the time for the cavity to collapse. The integration can be performed in terms of cavity radius by incorporating Eq. (9), and it is found that the value of the integral is exceedingly small, varying between 10^{-7} at small values of k to 10^{-4} at $k=20$.

It seems reasonable to conclude that a more accurate calculation of the integral would have resulted in a value of zero for all k, thus confirming the idea that p_B, the Borda-Carnot recovery pressure, is the mean pressure at the downstream station.

Comparison with measurements

The calculation of the cavity collapse time using Rayleigh's result (Eq. 10) would be reasonably accurate if the duct parameter k $(4\pi\ell_a R_m/A_2)$ were less than unity; above this value, the error becomes rapidly larger. The critical cavity radius (R_m) is surprisingly small; for example,[2] taking the duct cross section A_2 to be 800 mm^2 and the value of ℓ_a to be equivalent to the mean cavity length (about 90 mm), the critical cavity radius is approximately 0.7 mm. Thus Rayleigh's result would be applicable only for cavities smaller than about 1 mm diameter. The initial diameter of cavities in this type of duct cavitation is generally considerably larger than this, around 15 to 20 mm. The value of k is therefore between about 10 and 20, giving a cavity collapse time of between three and four times the corresponding Rayleigh cavity collapse time.

Detailed measurements of maximum, that is, initial, cavity radius, collapse time and mean attached cavity length are available in Lush and Peters[1] and Lush and Skipp.[2] These measurements were made by taking cine films of the cavitation at 3000 frames/s and then analyzing the cavity motion frame by frame. Length measurements were made by projecting the images onto a screen, and accurate assessment of time was obtained from a timing light generator. The time history of individual cavities could be followed easily and, since their behavior was not periodic, it was necessary to obtain averages of the various quantities over many cycles.

Taking the mean cavity length λ to be equivalent to the value of ℓ_a, the parameter k can be calculated (Table 1), and the ratio of collapse time to Rayleigh collapse time can be found from Figure 2. The Rayleigh collapse time, given by Eq. (10), can be calculated assuming the effective pressure at infinity is the

Table 1

	σ	λ (mm)	R_m (mm)	Rayl. t_c (ms)	Meas. t_c (ms)	Pred. t_c (ms)	Ratio
Convergent-divergent wedge	0.0033	87.0	8.61	0.64	2.21	2.05	1.08
$U_1 = 25.0$ m/s	0.013	93.6	7.78	0.56	1.55	1.78	0.87
Symmetrical wedge	0.025	91.2	8.11	0.73	2.52	2.32	1.09
$U_1 = 21.7$ m/s	0.05	63.3	7.83	0.69	1.71	1.86	0.92
	0.117	48.2	8.94	0.72	1.90	1.83	1.04
	0.2	34.6	8.91	0.67	2.75	1.49	1.85
	0.36	29.4	7.83	0.58	1.82	1.15	1.58

Borda-Carnot recovery pressure given by Eq. (5). Hence a prediction for actual collapse time can be made and compared with measurement. From Table 1, it can be seen that the prediction is good except for the symmetrical wedge inducer at high cavitation number. In the latter case, the cavitation resembles a vortex street, and thus it is likely that the cavity collapse is influenced significantly by the rotation of the liquid surrounding the cavity.

Although the source strength $\dot{V}$ is varying, the time-averaged source (or sink) strength can be calculated simply by dividing the maximum bubble volume by the collapse time. Values derived from measurement are between 1 and 1.5 dm^3/s, being between 10% and 20% of the total volumetric flow rate. The occurrence of these relatively large fractions indicates a source–sink pair model is required if the flow rate is to be maintained constant. Further justification for the neglect of the term in $U_c\dot{V}/A_2$ from Eq. (6) can be obtained from these measurements. The relative velocity between the cavities was found to be between 4 and 8 m/s, and since A_2 is 8 cm^2, $U_c\dot{V}/A_2$ is about 10 m^2/s^2. From Eq. (6), it can be argued that the inertia term is of order $(p_B - p_v)/\rho$, that is, about 120 m^2/s^2 in this case. Thus the term involving $\dot{V}$ is less than one tenth the other term. Further, the convection velocity of the detached cavity is found to be constant and about one half the throat velocity, and since the blockage is about 50%, the local fluid velocity and convection velocity are more or less equal; thus virtual mass effects should be small except at the moment of complete collapse.

Conclusions

A model of the dynamics of cavities associated with cavitation in a duct flow has been presented. The model yields a good prediction of cavity collapse time for a given maximum or initial cavity radius if the average separation between the effective source and sink is taken to be equivalent to the mean length of the attached cavity. The model also gives a time-averaged pressure downstream of the cavities that conforms to the Borda-Carnot recovery pressure, that is, the same value as achieved in steady flow. Finally, when the cavity collapses below a certain (fairly small) radius, typically about 0.5 to 1 mm, the Rayleigh solution is recovered.

Acknowledgment

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